

Divisibility by 2 of Stirling numbers of the second kind and their differences

Jianrong Zhao

*School of Economic Mathematics, Southwestern University of
Finance and Economics, Chengdu 610074, P.R. China*
Email: mathzjr@foxmail.com

Shaofang Hong*

Yangtze Center of Mathematics, Sichuan University, Chengdu 610064, P.R. China
Email: sfhong@scu.edu.cn, s-f.hong@tom.com, hongsf02@yahoo.com

Wei Zhao

Mathematical College, Sichuan University, Chengdu 610064, P.R. China

Abstract. Let n, k, a and c be positive integers and b be a nonnegative integer. Let $v_2(k)$ and $s_2(k)$ be the 2-adic valuation of k and the sum of binary digits of k , respectively. Let $S(n, k)$ be the Stirling number of the second kind. We first show that $v_2(S(c2^n, b2^{n+1} + a)) \geq s_2(a) - 1$, where $0 < a < 2^{n+1}$ and $2 \nmid c$. Further, we prove that $v_2(S(c2^n, (c-1)2^n + a)) = s_2(a) - 1$, where $n \geq 2$, $1 \leq a \leq 2^n$ and $2 \nmid c$. Finally, we show that if $3 \leq k \leq 2^n$ and k is not a power 2 minus 1, then $v_2(S(a2^n, k) - S(b2^n, k)) = n + v_2(a - b) - \lceil \log_2 k \rceil + s_2(k) + \delta(k)$, where $\delta(4) = 2$, $\delta(k) = 1$ if $k > 4$ is a power of 2, and $\delta(k) = 0$ otherwise. This confirms a conjecture of Lengyel raised in 2009 except that k is a power of 2 minus 1.

Keywords: Stirling numbers of the second kind, Congruences for power series, Bell polynomial, 2-Adic valuation

MR(2000) Subject Classification: Primary 11B73, 11A07

1 Introduction and the statements of main results

The Stirling numbers of the second kind $S(n, k)$ is defined for $n \in \mathbb{N}$ and positive integer $k \leq n$ as the number of ways to partition a set of n elements into exactly k non-empty subsets. It satisfies the recurrence relation

$$S(n, k) = S(n-1, k-1) + kS(n-1, k),$$

*Corresponding Author. The research of Hong was supported partially by National Science Foundation of China Grant # 10971145 and by the Ph.D. Programs Foundation of Ministry of Education of China Grant #20100181110073

with initial condition $S(0, 0) = 1$ and $S(n, 0) = 0$ for $n > 0$. There is also an explicit formula in terms of binomial coefficients given by

$$S(n, k) = \frac{1}{k!} \sum_{i=0}^k (-1)^i \binom{k}{i} (k-i)^n. \quad (1)$$

Divisibility properties of Stirling numbers have been studied from a number of different perspectives. It is known that for each fixed k , the sequence $\{S(n, k), n \geq k\}$ is periodic modulo prime powers. The length of this period has been studied by Carlitz [4] and Kwong [15]. Chan and Manna [5] characterized $S(n, k)$ modulo prime powers in terms of binomial coefficients.

Divisibility properties of integer sequences are often expressed in terms of p -adic valuations. Given a prime p and a positive integer m , there exist unique integers a and n , with $p \nmid a$ and $n \geq 0$, such that $m = ap^n$. The number n is called the p -adic valuation of m , denoted by $n = v_p(m)$. The numbers $\min\{v_p(k!S(n, k)) : m \leq k \leq n\}$ are important in algebraic topology, see, for example, [3, 9, 10, 11, 19, 20]. Some work on evaluating $v_p(k!S(n, k))$ has appeared in above papers as well as in [6, 8, 20, 22].

In this paper, we concern on the 2-adic valuations of the Stirling numbers of the second kind. Lengyel [16] studied the 2-adic valuations of $S(n, k)$ and conjectured, proved by Wannemacker [21], $v_2(S(2^n, k)) = s_2(k) - 1$, where $s_2(k)$ means the base 2 digital sum of k . Lengyel [17] showed that if $1 \leq k \leq 2^n$, then $v_2(S(c2^n, k)) = s_2(k) - 1$ for any positive integer c . Meanwhile, Lengyel [17] proved that $v_2(S(c2^n, k)) \geq s_2(k) - 1$ if $c \geq 1$ be an odd integer and $1 \leq k \leq 2^{n+1}$. We here show that a more general result true. That is, we have

Theorem 1.1 *Let $n, a, b, c \in \mathbb{N}$ with $0 < a < 2^{n+1}$, $b2^{n+1} + a \leq c2^n$ and $c \geq 1$ being odd. Then*

$$v_2(S(c2^n, b2^{n+1} + a)) \geq s_2(a) - 1.$$

If one picks $b = \frac{c-1}{2}$ and $1 \leq a \leq 2^n$, then the lower bound in Theorem 1.1 is arrived as the following result shows.

Theorem 1.2 *Let $a, c, n \in \mathbb{N}$ with $c \geq 1$ being odd, $n \geq 2$ and $1 \leq a \leq 2^n$. Then*

$$v_2(S(c2^n, (c-1)2^n + a)) = s_2(a) - 1.$$

Another interesting property is related to the difference of Stirling numbers of the second kind. Lengyel [17] studied the 2-adic valuations of the difference $S(c2^{n+1}, k) - S(c2^n, k)$ with $1 \leq k \leq 2^n$ and $c \geq 1$ odd. In the meantime, Lengyel posed the following conjecture.

Conjecture 1.1. [17] *Let $n, k, a, b \in \mathbb{N}$, $c \geq 1$ being odd and $3 \leq k \leq 2^n$. Then*

$$v_2(S(c2^{n+1}, k) - S(c2^n, k)) = n + 1 - f(k)$$

and

$$v_2(S(a2^n, k) - S(b2^n, k)) = n + 1 + v_2(a - b) - f(k)$$

for some function $f(k)$ which is independent of n .

Note that Lengyel [17] proved that Conjecture 1.1 is true for any integer k with $s_2(k) \leq 2$. As usual, for any real number x , we let $\lceil x \rceil$ and $\lfloor x \rfloor$ denote the smallest integer no less than x and the biggest integer no more than x , respectively. We have the following result.

Theorem 1.3 *Let $n, k, a, b \in \mathbb{N}$, $c \geq 1$ being odd, $3 \leq k \leq 2^n$, and $a > b$. If k is not a power of 2 minus 1, then*

$$v_2(S(a2^n, k) - S(b2^n, k)) = n + v_2(a - b) - \lceil \log_2 k \rceil + s_2(k) + \delta(k), \quad (2)$$

where $\delta(4) = 2$, $\delta(k) = 1$ if $k > 4$ is a power of 2, and $\delta(k) = 0$ otherwise. In particular,

$$v_2(S(c2^{n+1}, k) - S(c2^n, k)) = n - \lceil \log_2 k \rceil + s_2(k) + \delta(k). \quad (3)$$

By Theorem 1.3, we know that Conjecture 1.1 is true except that k is a power of 2 minus 1. At present, we are unable to prove Conjecture 1.1 for the remaining case that k is a power of 2 minus 1 because we encounter difficulties in strengthening the Junod's congruence about the Bell polynomials [13].

In order to prove Theorem 1.3, we will need a special case of the 2-adic valuation of $S(n, k)$, which can be stated as follows.

Theorem 1.4 *Let $a, b, c, m, n \in \mathbb{Z}^+$, $1 \leq a < 2^{n+1}$, $m \geq n + 2 + \lceil \log_2 b \rceil$ and $c \geq 1$ being odd. Then*

$$v_2(S(c2^m + b2^{n+1} + 2^n, b2^{n+2} + a)) \begin{cases} = n, & \text{if } a = 2^{n+1} - 1, \\ \geq s_2(a), & \text{if } a < 2^{n+1} - 1. \end{cases}$$

This paper is organized as follows. In Section 2, we present some preliminary results. Then in Section 3, we give the proofs of Theorems 1.1 and 1.2. Consequently, in Section 4, we show Theorem 1.4. Finally, in Section 5, we use Theorems 1.1 and 1.4 to show Theorem 1.3.

2 Preliminary lemmas

In this section, we give several known results which are needed for the proof of our main results. The first two results are well known.

Lemma 2.1 (*Legendre*) *Let $n \in \mathbb{N}$. Then $v_2(n!) = n - s_2(n)$.*

Lemma 2.2 [14] (Kummer) Let k and $n \in \mathbb{N}$ be such that $k \leq n$. Then $v_2(\binom{n}{k}) = s_2(k) + s_2(n - k) - s_2(n)$. Moreover, $s_2(k) + s_2(n - k) \geq s_2(n)$.

Lemma 2.3 [17] Let $k, n, c \in \mathbb{N}$ and $1 \leq k \leq 2^n$. Then $v_2(S(c2^n, k)) = s_2(k) - 1$.

Lemma 2.4 [17] Let $k, n, c \in \mathbb{N}$, $2^n < k < 2^{n+1} - 1$ and $c \geq 3$ be an odd integer. Then $v_2(S(c2^n, k)) \geq s_2(k)$ and $v_2(S(c2^n, 2^{n+1} - 1)) = n$.

Lemma 2.5 [17] Let $m, n, c \in \mathbb{N}$ and $0 \leq m < n$. Then $v_2(S(c2^n + 2^m, 2^n)) = n - 1 - m$.

Lemma 2.6 [21] Let $k, n, m \in \mathbb{N}$ and $0 \leq k \leq n + m$. Then

$$S(n + m, k) = \sum_{j=1}^k \sum_{i=0}^j \binom{j}{i} \frac{(k-i)!}{(k-j)!} S(n, k-i) S(m, j).$$

Lemma 2.7 [1] For $r \geq \max(k_1, k_2) + 2$, we have

$$\frac{k_1!k_2!(r-1)!}{(k_1+k_2+1)!} S(k_1+k_2+2, r) = \sum_{i=1}^{r-1} (i-1)!(r-i-1)! S(k_1+1, i) S(k_2+1, r-i).$$

Lemma 2.8 [13] Let $m, n, v \in \mathbb{N}$, $v \geq 1$ and p be a prime number. Then

$$B_{m+np^v}(x) \equiv \sum_{j=0}^n \binom{n}{j} (x^p + x^{p^2} + \cdots + x^{p^v})^{n-j} B_{m+j}(x) \pmod{\frac{np}{2}\mathbb{Z}_p[x]}, \quad (4)$$

where the Bell polynomials are defined by

$$B_n(x) = \sum_{k=0}^n S(n, k) x^k, \quad n \geq 0. \quad (5)$$

Let $n = \sum_{\lambda=0}^{\infty} \varepsilon_{\lambda}(n) 2^{\lambda}$ with $\varepsilon_{\lambda}(n) \in \{0, 1\}$. Then $s_2(n) = \sum_{\lambda=0}^{\infty} \varepsilon_{\lambda}(n)$. Further, we have the following result.

Lemma 2.9 Let m and $n \in \mathbb{N}$. Then $s_2(m+n) = s_2(m) + s_2(n)$ if and only if $\varepsilon_{\lambda}(m) + \varepsilon_{\lambda}(n) = \varepsilon_{\lambda}(m+n)$ for all $\lambda \in \mathbb{N}$.

Proof. This lemma follows immediately from the proof of Lemma 1 in [21]. \square

Lemma 2.10 Let $n, a \in \mathbb{N}$ and $1 \leq a < 2^{n+1}$. Define the set J of positive integers by $J := \{1 \leq j \leq 2^n \mid s_2(2^{n+1} + a - j) + s_2(j) = s_2(2^{n+1} + a)\}$. Then $|J| = 2^{s_2(a)} - 1$ if $1 \leq a \leq 2^n$, and $|J| = 2^{s_2(a)-1}$ if $2^n < a < 2^{n+1}$.

Proof. For any positive integer d , define $M_d := \{\lambda \in \mathbb{N} \mid \varepsilon_\lambda(d) = 1\}$. Then $d = \sum_{\lambda \in M_d} 2^\lambda$ and $s_2(d) = |M_d|$. By Lemma 2.9 we know that $s_2(2^{n+1} + a - j) + s_2(j) = s_2(2^{n+1} + a)$ if and only if

$$\varepsilon_\lambda(j) + \varepsilon_\lambda(2^{n+1} + a - j) = \varepsilon_\lambda(2^{n+1} + a) \quad (6)$$

for all $\lambda \in \mathbb{N}$. Therefore for any given $\lambda \in \mathbb{N}$, $\varepsilon_\lambda(j) = 0$ or 1 if $\varepsilon_\lambda(2^{n+1} + a) = 1$, and $\varepsilon_\lambda(j) = 0$ if $\varepsilon_\lambda(2^{n+1} + a) = 0$. It then follows that for any given integer $1 \leq a \leq 2^n$, $j \in J$ if and only if $\emptyset \neq M_j \subseteq M_a$. So $|J| = 2^{|M_a|} - 1 = 2^{s_2(a)} - 1$ if $1 \leq a \leq 2^n$.

Now let $2^n < a < 2^{n+1}$. So if $j = 2^n$, then one can check that $s_2(2^{n+1} + a - 2^n) + s_2(2^n) = s_2(2^{n+1} + a)$. This implies that $2^n \in J$. On the other hand, since $1 < a - 2^n < 2^n$, we get that $j \in J \setminus \{2^n\}$ if and only if $\emptyset \neq M_j \subseteq M_{a-2^n}$. Hence $|J| = 2^{|M_{a-2^n}|} = 2^{s_2(a)-1}$ if $2^n < a < 2^{n+1}$. The proof of Lemma 2.10 is complete. \square

Lemma 2.11 *Let $n, a, c \in \mathbb{N}$ with $c \geq 1$ being odd and $1 \leq a \leq 2^n$. Then*

$$s_2(c2^n - a) = s_2(c) + n - v_2(a) - s_2(a). \quad (7)$$

Proof. If $a = 2^n$, then it is easy to check that (7) is true. Now let $1 \leq a < 2^n$. We can write $a = \sum_{\lambda=v_2(a)}^{n-1} \varepsilon_\lambda(a) 2^\lambda$. Clearly $s_2(a) = \sum_{\lambda=v_2(a)}^{n-1} \varepsilon_\lambda(a)$ and $\varepsilon_{v_2(a)}(a) = 1$. Then

$$\begin{aligned} c2^n - a &= (c-1)2^n + 2^n - a \\ &= (c-1)2^n + \left(2^{v_2(a)} + \sum_{\lambda=v_2(a)}^{n-1} 2^\lambda\right) - \sum_{\lambda=v_2(a)}^{n-1} \varepsilon_\lambda(a) 2^\lambda \\ &= (c-1)2^n + \sum_{\lambda=v_2(a)}^{n-1} (1 - \varepsilon_\lambda(a)) 2^\lambda + 2^{v_2(a)}. \end{aligned} \quad (8)$$

Since $s_2(c-1) = s_2(c) - 1$, by (8) we have

$$\begin{aligned} s_2(c2^n - a) &= s_2(c-1) + \sum_{\lambda=v_2(a)}^{n-1} (1 - \varepsilon_\lambda(a)) + 1 \\ &= s_2(c) + \sum_{\lambda=v_2(a)}^{n-1} (1 - \varepsilon_\lambda(a)) \\ &= s_2(c) + n - v_2(a) - s_2(a) \end{aligned}$$

as required. This completes the proof of Lemma 2.11. \square

Lemma 2.12 [12] *Let $N \geq 2$ be an integer and r, t be odd numbers. For any $m \in \mathbb{Z}^+$, we have $v_2((r2^N - 1)^{t2^m} - 1) = m + N$.*

3 Proofs of Theorems 1.1 and 1.2

In this section, we show Theorems 1.1 and 1.2. We begin with the proof of Theorem 1.1.

Proof of Theorem 1.1. If $b = 0$, then Theorem 1.1 is true by Lemmas 2.3 and 2.4. In what follows we let $b \geq 1$. There exists a unique integer $e \geq 0$ such that $2^e \leq b < 2^{e+1}$. We proceed with induction on e . First we consider the case $e = 0$, i.e., $b = 1$. Using Lemma 2.6 with n, m and k replaced by $(c-1)2^n, 2^n$ and $2^{n+1} + a$, respectively, we have

$$S(c2^n, 2^{n+1} + a) = \sum_{j=1}^{2^{n+1}+a} \sum_{i=0}^j f(i, j) = \sum_{j=1}^{2^n} \sum_{i=0}^{2^n} f(i, j), \quad (9)$$

where

$$f(i, j) := \binom{j}{i} \frac{(2^{n+1} + a - i)!}{(2^{n+1} + a - j)!} S((c-1)2^n, 2^{n+1} + a - i) S(2^n, j).$$

Since c is an odd integer, $v_2((c-1)2^n) \geq n+1$. It then follows from Lemmas 2.1, 2.3 and 2.4 that

$$\begin{aligned} v_2(f(i, j)) &\geq v_2\left(\frac{(2^{n+1} + a - i)!}{(2^{n+1} + a - j)!}\right) + v_2(S((c-1)2^n, 2^{n+1} + a - i)) + v_2(S(2^n, j)) \\ &\geq v_2((2^{n+1} + a - i)!) - v_2((2^{n+1} + a - j)!) + s_2(2^{n+1} + a - i) - 1 + s_2(j) - 1 \\ &= (j - i) + s_2(2^{n+1} + a - j) - s_2(2^{n+1} + a - i) + s_2(2^{n+1} + a - i) + s_2(j) - 2 \\ &\geq s_2(2^{n+1} + a - j) + s_2(j) - 2 \end{aligned} \quad (10)$$

since $j \geq i$. By Lemma 2.2 we know that

$$s_2(j) + s_2(2^{n+1} + a - j) \geq s_2(2^{n+1} + a).$$

So by (10) and noting that $0 < a < 2^{n+1}$, we obtain

$$v_2(f(i, j)) \geq s_2(2^{n+1} + a) - 2 = s_2(a) - 1. \quad (11)$$

It then follows from (9) and (11) that

$$v_2(S(c2^n, 2^{n+1} + a)) \geq \min_{0 \leq i \leq j \leq 2^n} \{v_2(f(i, j))\} \geq s_2(a) - 1.$$

Hence Theorem 1.1 is true if $e = 0$. In what follows, we let $e \geq 1$.

Assume that Theorem 3.1 is true for the case t with $t \leq e-1$. Then $v_2(S(c2^n, b2^{n+1} + a)) \geq s_2(a) - 1$ for any integers b with $0 \leq b < 2^e$. In the following we prove that Theorem 1.1 is true for the case e . This is equivalent to showing Theorem 1.1 for all integers $b \in [2^e, 2^{e+1})$.

Let $b \in [2^e, 2^{e+1})$ be any given integer. Since $c2^n \geq b2^{n+1} + a$, there exist two positive integers c_1 and c_2 such that $c = c_1 + c_2 2^{v_2(b)+1}$ and $c_1 < 2^{v_2(b)+1}$. So by Lemma 2.6 we have

$$S(c2^n, b2^{n+1} + a) = \sum_{j=1}^{c_1 2^n} \sum_{i=0}^j g(i, j), \quad (12)$$

where

$$g(i, j) := \binom{j}{i} \frac{(b2^{n+1} + a - i)!}{(b2^{n+1} + a - j)!} S(c_2 2^{n+v_2(b)+1}, b2^{n+1} + a - i) S(c_1 2^n, j).$$

We first claim that

$$v_2(S(c_2 2^{n+v_2(b)+1}, b2^{n+1} + a - i)) \geq s_2(b2^{n+1} + a - i) - s_2(b). \quad (13)$$

If $v_2(c_2) + v_2(b) \geq e$, then $b2^{n+1} + a - i < 2^{e+n+2} \leq 2^{v_2(b)+v_2(c_2)+n+2}$ since $a < 2^{n+1}$ and $2^e \leq b < 2^{e+1}$. So by Lemmas 2.3 and 2.4, we obtain that

$$\begin{aligned} & v_2(S(c_2 2^{n+v_2(b)+1}, b2^{n+1} + a - i)) \\ &= v_2\left(S\left(\frac{c_2}{2^{v_2(c_2)}} 2^{n+v_2(b)+v_2(c_2)+1}, b2^{n+1} + a - i\right)\right) \\ &\geq s_2(b2^{n+1} + a - i) - 1 \\ &\geq s_2(b2^{n+1} + a - i) - s_2(b) \end{aligned}$$

as desired. The claim (13) is proved in this case.

If $v_2(c_2) + v_2(b) \leq e - 1$, then we can write $b = b_1 2^{v_2(c_2)+v_2(b)+1} + b_2$ for some integers $0 < b_1 < 2^{e-v_2(c_2)-v_2(b)}$ and $2^{v_2(b)} \leq b_2 < 2^{v_2(c_2)+v_2(b)+1}$ since $2^e \leq b < 2^{e+1}$. One can deduce that $s_2(b2^{n+1} + a - i) = s_2(b_2 2^{n+1} + a - i) + s_2(b_1)$. It then follows from the inductive hypothesis that

$$\begin{aligned} & v_2(S(c_2 2^{n+v_2(b)+1}, b2^{n+1} + a - i)) \\ &= v_2\left(S\left(\frac{c_2}{2^{v_2(c_2)}} 2^{n+v_2(b)+v_2(c_2)+1}, b_1 2^{n+v_2(b)+v_2(c_2)+2} + b_2 2^{n+1} + a - i\right)\right) \\ &\geq s_2(b_2 2^{n+1} + a - i) - 1 \\ &= s_2(b2^{n+1} + a - i) - s_2(b_1) - 1 \\ &\geq s_2(b2^{n+1} + a - i) - s_2(b) \end{aligned}$$

as required. So the claim (13) is true for this case. Hence the claim (13) is proved.

Consequently, we claim that for all the integers i and j such that $0 \leq i \leq j \leq c_1 2^n$ with $c_1 < 2^{v_2(b)+1}$,

$$v_2(g(i, j)) \geq s_2(a) - 1. \quad (14)$$

From (12) and the claim (14) we deduce that

$$v_2(S(c2^n, b2^{n+1} + a)) \geq \min_{0 \leq i \leq j \leq c_1 2^n} \{v_2(g(i, j))\} \geq s_2(a) - 1.$$

In other words, Theorem 1.1 holds if $b \in [2^e, 2^{e+1})$. It remains to show that (14) is true which will be done in the following.

If $1 \leq j < 2^{n+1}$, then by Lemmas 2.3 and 2.4 we have $v_2(S(c_1 2^n, j)) \geq s_2(j) - 1$. Thus using Lemmas 2.1-2.2 and the claim (13), we derive from $a < 2^{n+1}$ that

$$\begin{aligned}
v_2(g(i, j)) &\geq v_2\left(\frac{(b2^{n+1} + a - i)!}{(b2^{n+1} + a - j)!}\right) + s_2(b2^{n+1} + a - i) - s_2(b) + s_2(j) - 1 \\
&\geq s_2(b2^{n+1} + a - j) - s_2(b2^{n+1} + a - i) + s_2(b2^{n+1} + a - i) - s_2(b) + s_2(j) - 1 \\
&\geq s_2(b2^{n+1} + a - j) + s_2(j) - s_2(b) - 1 \\
&\geq s_2(b2^{n+1} + a) - s_2(b) - 1 \\
&= s_2(a) - 1
\end{aligned}$$

as required. Hence the claim (14) is true in this case.

If $2^{n+1} \leq j \leq c_1 2^n$, then we may let $j = j_1 2^{n+1} + j_2$ for some integers $0 \leq j_2 < 2^{n+1}$ and $j_1 < 2^{v_2(b)}$ since $c_1 < 2^{v_2(b)+1}$. If $j_2 = 0$, i.e., $j = j_1 2^{n+1}$, then by the claim (13) and Lemmas 2.1-2.2, noting that $a < 2^{n+1}$, we get that

$$\begin{aligned}
v_2(g(i, j)) &\geq v_2\left(\frac{(b2^{n+1} + a - i)!}{(b2^{n+1} + a - j)!}\right) + v_2(S(c_2 2^{n+v_2(b)+1}, b2^{n+1} + a - i)) \\
&\geq s_2(b2^{n+1} + a - j) - s_2(b2^{n+1} + a - i) + s_2(b2^{n+1} + a - i) - s_2(b) \\
&= s_2((b - j_1)2^{n+1} + a) - s_2(b) \\
&= s_2(b - j_1) + s_2(a) - s_2(b) \\
&\geq s_2(a)
\end{aligned}$$

since $j_1 < 2^{v_2(b)}$ implying that $s_2(b - j_1) \geq s_2(b)$. Hence the claim (14) is true if $j_2 = 0$. Now let $j_2 \geq 1$. Since $j_1 < 2^{v_2(b)} \leq 2^e$, by the inductive hypothesis we have

$$v_2(S(c_1 2^n, j)) = v_2(S(c_1 2^n, j_1 2^{n+1} + j_2)) \geq s_2(j_2) - 1. \quad (15)$$

Thus by Lemmas 2.1-2.2, (13) and (15) we obtain

$$\begin{aligned}
v_2(g(i, j)) &\geq v_2\left(\frac{(b2^{n+1} + a - i)!}{(b2^{n+1} + a - j)!}\right) + v_2(S(c_2 2^{n+v_2(b)+1}, b2^{n+1} + a - i)) + S(c_1 2^n, j) \\
&\geq s_2(b2^{n+1} + a - j) - s_2(b2^{n+1} + a - i) + s_2(b2^{n+1} + a - i) - s_2(b) + s_2(j_2) - 1 \\
&= s_2(b2^{n+1} + a - j) + s_2(j_2) - s_2(b) - 1 \\
&= s_2((b - j_1)2^{n+1} + a - j_2) + s_2(j_2) - s_2(b) - 1 \\
&\geq s_2((b - j_1)2^{n+1} + a) - s_2(b) - 1 \\
&= s_2(b - j_1) + s_2(a) - s_2(b) - 1 \\
&\geq s_2(a) - 1
\end{aligned}$$

since $s_2(b - j_1) \geq s_2(b)$. Hence the claim (14) holds if $j_2 \geq 1$. So the claim (14) is proved.

This completes the proof of Theorem 1.1. \square

Consequently, we turn our attention to the proof of Theorem 1.2.

Proof of Theorem 1.2. If $a = 2^n$, then by definition of Stirling numbers of the second kind, we have

$$S(c2^n, (c-1)2^n + a) = S(c2^n, c2^n) = 1.$$

This implies that $v_2(S(c2^n, c2^n)) = s_2(2^n) - 1$. So Theorem 1.2 is true in this case.

Now let $1 \leq a < 2^n$ and $b = \frac{c-1}{2}$. Then

$$S(c2^n, (c-1)2^n + a) = S(b2^{n+1} + 2^n, b2^{n+1} + a).$$

To prove Theorem 1.2, it is sufficient to show that

$$v_2(S(b2^{n+1} + 2^n, b2^{n+1} + a)) = s_2(a) - 1. \quad (16)$$

For $t \in \mathbb{N}$, define

$$A_t := \{b \in \mathbb{N} \mid s_2(b) = t\}. \quad (17)$$

Then $\mathbb{N} = \bigcup_{t=0}^{\infty} A_t$. We proceed with induction on t . First we consider the case $t = 0$. If $b \in A_0$, then $b = 0$. By Lemma 2.3 we have

$$v_2(S(b2^{n+1} + 2^n, b2^{n+1} + a)) = v_2(S(2^n, a)) = s_2(a) - 1.$$

So Theorem 1.2 holds if $t = 0$.

Assume that Theorem 1.2 is true for the case r with $r \leq t-1$. Hence (16) holds for any positive integers $b \in A_0 \cup A_1 \cup \dots \cup A_{t-1}$. In the following let $t \geq 1$ and we will prove that Theorem 1.2 is true for the case t , which is equivalent to showing (16) for all the integers $b \in A_t$.

Let $b \in A_t$ be a given integer. We first notice that

$$b2^{n+1} + a \geq \max(b2^{n+1} - 1, 2^n - 1) + 2.$$

Letting $k_1 = b2^{n+1} - 1$, $k_2 = 2^n - 1$ and $r = b2^{n+1} + a$ in Lemma 2.7 gives us that

$$\begin{aligned} & \frac{(b2^{n+1} - 1)!(2^n - 1)!}{(b2^{n+1} + 2^n - 1)!} (b2^{n+1} + a - 1)! S(b2^{n+1} + 2^n, b2^{n+1} + a) \\ &= \sum_{i=1}^{b2^{n+1} + a - 1} (i-1)! (b2^{n+1} + a - i - 1)! S(2^n, i) S(b2^{n+1}, b2^{n+1} + a - i) \\ &= \sum_{i=a}^{2^n} \frac{1}{i(b2^{n+1} + a - i)} i! S(2^n, i) (b2^{n+1} + a - i)! S(b2^{n+1}, b2^{n+1} + a - i). \end{aligned}$$

It follows that

$$(b2^{n+1} + a)!S(b2^{n+1} + 2^n, b2^{n+1} + a) = \frac{(b2^{n+1} + 2^n - 1)!}{(b2^{n+1} - 1)!(2^n - 1)!} \sum_{i=a}^{2^n} l(i), \quad (18)$$

where

$$l(i) := \frac{b2^{n+1} + a}{i(b2^{n+1} + a - i)} i!S(2^n, i)(b2^{n+1} + a - i)!S(b2^{n+1}, b2^{n+1} + a - i).$$

Write $b = (2b_0 + 1)2^{v_2(b)}$ for some $b_0 \in \mathbb{N}$. Clearly $s_2(b_0) = s_2(b) - 1 = t - 1$ since $b \in A_t$. Then $b_0 \in A_{t-1}$. It then follows from Lemma 2.1 that

$$\begin{aligned} v_2\left(\frac{(b2^{n+1} + 2^n - 1)!}{(b2^{n+1} - 1)!(2^n - 1)!}\right) &= v_2((b2^{n+1} + 2^n - 1)!) - v_2((b2^{n+1} - 1)!) - v_2((2^n - 1)!) \\ &= 1 - s_2(b2^{n+1} + 2^n - 1) + s_2(b2^{n+1} - 1) + s_2(2^n - 1) \\ &= 1 - s_2(b2^{n+1}) + s_2(b_0 2^{n+v_2(b)+2} + 2^{n+v_2(b)+1} - 1) \\ &= 1 - s_2(b) + s_2(b_0) + n + v_2(b) + 1 \\ &= n + v_2(b) + 1. \end{aligned} \quad (19)$$

On the other hand, we have

$$\begin{aligned} v_2((b2^{n+1} + a)!) &= (b2^{n+1} + a) - s_2((b2^{n+1} + a)) \\ &= b2^{n+1} + a - s_2(b) - s_2(a). \end{aligned} \quad (20)$$

In order to show that (16) is true, by (18)-(20) we only need to show that

$$v_2\left(\sum_{i=a}^{2^n} l(i)\right) = \min_{a \leq i \leq 2^n} \{v_2(l(i))\} = (b2^{n+1} + a) - (s_2(b) + v_2(b) + n + 2). \quad (21)$$

In the following we discuss the 2-adic valuation of $l(i)$ with $a \leq i \leq 2^n$.

Since $b_0 \in A_{t-1}$ and $0 < 2^{n+v_2(b)+1} + a - i \leq 2^{n+v_2(b)+1}$, by the inductive hypothesis and Lemma 2.3, we can derive that

$$\begin{aligned} &v_2(S(b2^{n+1}, b2^{n+1} + a - i)) \\ &= v_2\left(S\left(b_0 2^{n+v_2(b)+2} + 2^{n+v_2(b)+1}, b_0 2^{n+v_2(b)+2} + 2^{n+v_2(b)+1} + a - i\right)\right) \\ &= s_2(2^{n+v_2(b)+1} + a - i) - 1 \\ &= s_2((2b_0 + 1)2^{n+v_2(b)+1} + a - i) - s_2(b_0) - 1 \\ &= s_2(b2^{n+1} + a - i) - s_2(b). \end{aligned} \quad (22)$$

Furthermore, by Lemmas 2.1, 2.3 and (22) we can compute that

$$\begin{aligned} &v_2(i!S(2^n, i)(b2^{n+1} + a - i)!S(b2^{n+1}, b2^{n+1} + a - i)) \\ &= i - s_2(i) + s_2(i) - 1 + (b2^{n+1} + a - i) - s_2(b2^{n+1} + a - i) + s_2(b2^{n+1} + a - i) - s_2(b) \\ &= (b2^{n+1} + a) - s_2(b) - 1. \end{aligned} \quad (23)$$

So by (23) we have

$$v_2(l(i)) = (b2^{n+1} + a) - s_2(b) - 1 + v_2(b2^{n+1} + a) - v_2(i) - v_2(b2^{n+1} + a - i). \quad (24)$$

If $i = a$, then by (24) and noticing that $a \leq 2^n$, we get

$$\begin{aligned} v_2(l(a)) &= (b2^{n+1} + a) - s_2(b) - 1 + v_2(b2^{n+1} + a) - v_2(a) - v_2(b2^{n+1}) \\ &= (b2^{n+1} + a) - (s_2(b) + v_2(b) + n + 2). \end{aligned} \quad (25)$$

If $a < i \leq 2^n$ and $v_2(i) \leq v_2(b2^{n+1} + a)$, then

$$v(i) - v_2(b2^{n+1} + a) + v_2(b2^{n+1} + a - i) \leq v_2(b2^{n+1} + a - i) < n. \quad (26)$$

It then follows from (24) and (26) that

$$v_2(l(i)) > (b2^{n+1} + a) - s_2(b) - 1 - n > (b2^{n+1} + a) - (s_2(b) + v_2(b) + n + 2). \quad (27)$$

If $a < i \leq 2^n$ and $v_2(i) > v_2(b2^{n+1} + a)$, then we have

$$v_2(i) - v_2(b2^{n+1} + a) + v_2(b2^{n+1} + a - i) = v_2(i) \leq n. \quad (28)$$

So by (24) and (28) we have

$$v_2(l(i)) \geq (b2^{n+1} + a) - s_2(b) - 1 - n > (b2^{n+1} + a) - (s_2(b) + v_2(b) + n + 2). \quad (29)$$

Thus the desired result (21) follows immediately from (25), (27) and (29). So (16) holds if $b \in A_t$, which implies that Theorem 1.2 is true if $b \in A_t$.

The proof of Theorem 1.2 is complete. \square

4 Proof of Theorem 1.4

In this section, we always let $a, b, c, m, n \in \mathbb{Z}^+$, $1 \leq a < 2^{n+1}$, $m \geq n + 2 + \lfloor \log_2 b \rfloor$ and $c \geq 1$ being odd. For any integers i and j with $0 \leq i \leq j \leq b2^{n+1} + 2^n$, we define

$$h(i, j) := \binom{j}{i} \frac{(b2^{n+2} + a - i)!}{(b2^{n+2} + a - j)!} S(c2^m, b2^{n+2} + a - i) S(b2^{n+1} + 2^n, j). \quad (30)$$

Let

$$\begin{aligned} \Delta_1 &:= \sum_{j=1}^{2^n} \sum_{i=0}^j h(i, j), & \Delta_2 &:= \sum_{j=2^{n+1}}^{2^{n+1}-2} \sum_{i=0}^j h(i, j), \\ \Delta_3 &:= \sum_{i=0}^{2^{n+1}-1} h(i, 2^{n+1} - 1), & \Delta_4 &:= \sum_{j=b2^{n+1}+1}^{b2^{n+1}+2^n} \sum_{i=0}^j h(i, j). \end{aligned} \quad (31)$$

We have the following Lemma.

Lemma 4.1 *Each of the following is true:*

- (i) For $l = 1$ and 4 , we have $v_2(\Delta_l) \begin{cases} = s_2(a) - 1, & \text{if } 1 \leq a \leq 2^n \text{ and } s_2(b) = 1, \\ \geq s_2(a), & \text{otherwise;} \end{cases}$
- (ii) $v_2(\Delta_2) \geq s_2(a)$;
- (iii) $v_2(\Delta_3) \begin{cases} = n, & \text{if } a = 2^{n+1} - 1 \text{ and } s_2(b) = 1, \\ \geq s_2(a), & \text{otherwise;} \end{cases}$
- (iv) $v_2(\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4) \begin{cases} = n, & \text{if } a = 2^{n+1} - 1 \text{ and } s_2(b) = 1, \\ \geq s_2(a), & \text{otherwise.} \end{cases}$

Proof. Evidently, part (iv) follows immediately from parts (i)-(iii). So we need only to show parts (i)-(iii) which will be done in what follows. By Lemmas 2.1 and 2.2, we have

$$v_2\left(\binom{j}{i} \frac{(b2^{n+2} + a - i)!}{(b2^{n+2} + a - j)!}\right) = s_2(i) + s_2(j - i) - s_2(j) + j - i + s_2(b2^{n+2} + a - j) - s_2(b2^{n+2} + a - i). \quad (32)$$

(i). First we treat with Δ_1 . Let $1 \leq j \leq 2^n$ and $0 \leq i \leq j$. By Lemma 2.3

$$v_2(S(b2^{n+1} + 2^n, j)) = s_2(j) - 1. \quad (33)$$

Let $m > n + 2 + \lfloor \log_2 b \rfloor$. Since $a < 2^{n+1}$ and $1 \leq i \leq 2^n$, we have $b2^{n+2} + a - i < 2^m$. By Lemma 2.3 we obtain $v_2(S(c2^m, b2^{n+2} + a - i)) = s_2(b2^{n+2} + a - i) - 1$. It then follows from (30), (32), (33) and Lemma 2.2, we obtain that

$$\begin{aligned} v_2(h(i, j)) &= s_2(i) + s_2(j - i) + j - i + s_2(b2^{n+2} + a - j) - 2 \\ &\geq s_2(j) + s_2(b2^{n+2} + a - j) + j - i - 2 \\ &\geq s_2((b2^{n+2} + a) - 2) \\ &\geq s_2(a) - 1, \end{aligned} \quad (34)$$

where equality holds if and only if $j = i$, $s_2(b) = 1$ and $s_2(b2^{n+2} + a - j) + s_2(j) = s_2(b2^{n+2} + a)$. So by (31) we get

$$\Delta_1 = 2^{s_2(a)} \tilde{\Delta}_1 + 2^{s_2(a)-1} \sum_{(i,j) \in \tilde{J}} \tilde{h}(i, j), \quad (35)$$

where $\tilde{\Delta}_1 \in \mathbb{Z}^+$ and $\tilde{J} := \{(i, j) \mid \tilde{h}(i, j) \text{ is odd}, 1 \leq i \leq j \leq 2^n\}$. Then

$$\begin{aligned} \tilde{J} &= \{(i, j) \mid j = i, s_2(b) = 1 \text{ and } s_2(b2^{n+2} + a - j) + s_2(j) = s_2(b2^{n+2} + a)\} \\ &= \{(j, j) \mid s_2(b) = 1 \text{ and } s_2(b2^{n+2} + a - j) + s_2(j) = s_2(b2^{n+2} + a)\} \\ &= \{1 \leq j \leq 2^n \mid s_2(b) = 1 \text{ and } s_2(2^{n+2} + a - j) + s_2(j) = s_2(2^{n+2} + a)\}. \end{aligned}$$

Thus by Lemma 2.10 we know that $|\tilde{J}| = 2^{s_2(a)} - 1$ if $1 \leq a \leq 2^n$ and $2^{s_2(a)-1}$ else.

Furthermore, by (35), we derive that $v_2(\Delta_1)$ equals $s_2(a) - 1$ if $s_2(b) = 1$ and $1 \leq a \leq 2^n$, and is greater than $s_2(a)$ otherwise. So Lemma 4.1 (i) is true if $l = 1$ and $m > n + 2 + \lfloor \log_2 b \rfloor$.

Now let $m = n + 2 + \lfloor \log_2 b \rfloor$. If either $2^n < a < 2^{n+1}$, or $1 \leq a \leq 2^n$ and $1 \leq i < a$, then one can check that the following is true:

$$2^m \leq b2^{n+2} < b2^{n+2} + a - i < b2^{n+2} + a \leq 2^{m+1} - 1.$$

So Lemma 2.4 implies that

$$v_2(S(c2^m, b2^{n+2} + a - i)) \geq s_2(b2^{n+2} + a - i). \quad (36)$$

Thus by Lemma 2.2, (30), (32), (33) and (36) we deduce that

$$\begin{aligned} v_2(h(i, j)) &\geq s_2(i) + s_2(j - i) + j - i + s_2(b2^{n+2} + a - j) - 1 \\ &\geq s_2(j) + s_2(b2^{n+2} + a - j) + j - i - 1 \\ &\geq s_2(b2^{n+2} + a) - 1 \\ &\geq s_2(a). \end{aligned} \quad (37)$$

If $1 \leq a \leq 2^n$ and $a \leq i \leq j$, then $b2^{n+2} + a - i \leq b2^{n+2} \leq 2^m$. Then by Lemma 2.3 we get $v_2(S(c2^m, b2^{n+2} + a - i)) = s_2(b2^{n+2} + a - i) - 1$. Hence by (32), (30) and Lemma 2.2, we have

$$\begin{aligned} v_2(h(i, j)) &= s_2(i) + s_2(j - i) + j - i + s_2(b2^{n+2} + a - j) - 2 \\ &\geq s_2(j) + s_2(b2^{n+2} + a - j) + j - i - 2 \\ &\geq s_2(b2^{n+2} + a) - 2 \\ &\geq s_2(a) - 1, \end{aligned} \quad (38)$$

with equality holding if and only if

$$j = i, \quad s_2(b) = 1 \text{ and } s_2(b2^{n+2} + a - j) + s_2(j) = s_2(b2^{n+2} + a). \quad (39)$$

Since $1 \leq j \leq 2^n$ and $a \leq i \leq j$, by Lemma 2.9 we know that (39) holds only when $i = j = a$ and $s_2(b) = 1$. It follows from (37) and (38) that $v_2(h(i, j)) \geq s_2(a)$ except for $i = j = a \in [1, 2^n]$ and $s_2(b) = 1$, in which case one has $v_2(h(a, a)) = s_2(a) - 1$. Then by (31), we have $v_2(\Delta_1) = s_2(a) - 1$ if $a \in [1, 2^n]$ and $s_2(b) = 1$, and $v_2(\Delta_1) \geq s_2(a)$ otherwise. Thus Lemma 4.1 (i) is true if $l = 1$ and $m = n + 2 + \lfloor \log_2 b \rfloor$. So the statement for Δ_1 is proved.

Now we handle Δ_4 . Note that $b2^{n+1} + 1 \leq j \leq b2^{n+1} + 2^n$ and $0 \leq i \leq j$. Let $j = b2^{n+1} + j_0$ for some integer $1 \leq j_0 \leq 2^n$. By Theorem 1.2 we have

$$v_2(S(b2^{n+1} + 2^n, j)) = v_2(S(b2^{n+1} + 2^n, b2^{n+1} + j_0)) = s_2(j_0) - 1. \quad (40)$$

Since $m \geq n + 2 + \lfloor \log_2 b \rfloor$, we have $b2^{n+2} + a - j < b^{n+1} + a < 2^m$. So by Lemmas 2.3 and 2.4 we get

$$v_2(S(b2^{n+1} + 2^n, b2^{n+2} + a - i)) \geq s_2(b2^{n+2} + a - i) - 1 \quad (41)$$

and

$$v_2(S(b2^{n+1} + 2^n, b2^{n+2} + a - j)) = s_2(b2^{n+2} + a - j) - 1 \quad (42)$$

So by (30), (32), (40)-(42) and Lemma 2.2 we obtain

$$\begin{aligned} v_2(h(i, j)) &\geq s_2(i) + s_2(j - i) - s_2(j) + j - i + s_2(b2^{n+2} + a - j) + s_2(j_0) - 2 \\ &\geq s_2(b2^{n+1} + a - j_0) + s_2(j_0) - 2 \\ &\geq s_2(a) - 1 \end{aligned} \quad (43)$$

where equality holds if and only if $j = i$, $s_2(b) = 1$ and $s_2(b2^{n+1} + a - j_0) + s_2(j_0) = s_2(b2^{n+1} + a)$. It is similar to Δ_1 with $m \geq n + 2 + \lfloor \log_2 b \rfloor$, by Lemma 2.10 we have $v_2(\Delta_4) = s_2(a) - 1$ if $a \in [1, 2^n]$ and $s_2(b) = 1$, and $v_2(\Delta_4) \geq s_2(a)$ otherwise. So Lemma 4.1 (i) is true if $l = 4$.

(ii). For Δ_2 , noticing that $2^n < j < 2^{n+1} - 1$ and $0 \leq i \leq j$, then by Lemmas 2.2-2.4, we get

$$v_2(S(c2^m, b2^{n+2} + a - i)S(b2^{n+1} + 2^n, j)) \geq s_2(b2^{n+2} + a - i) - 1 + s_2(j).$$

So by (30) and (32) we have

$$\begin{aligned} v_2(h(i, j)) &\geq s_2(i) + s_2(j - i) + j - i + s_2(b2^{n+2} + a - j) - 1 \\ &\geq s_2(b2^{n+2} + a) - 1 \\ &\geq s_2(a). \end{aligned} \quad (44)$$

Hence by (31) and (44), we have $v_2(\Delta_2) \geq s_2(a)$ as desired.

(iii). For Δ_3 , noting that $j = 2^{n+1} - 1$ and $0 \leq i \leq 2^{n+1} - 1$, it follows from Lemmas 2.2-2.4, (31) and (32) that

$$\begin{aligned} v_2(h(i, 2^{n+1} - 1)) &\geq s_2(i) + s_2(j - i) + j - i + s_2(b2^{n+2} + a - j) - 2 \\ &\geq s_2(b2^{n+2} + a - j) + s_2(j) - 2 \\ &\geq s_2(b2^{n+2} + a - 2^{n+1} + 1) + s_2(2^{n+1} - 1) - 2 \\ &= s_2(b2^{n+2} + a - 2^{n+1} + 1) + n - 1 \\ &\geq n, \end{aligned} \quad (45)$$

with equality holding if and only if $j = i = a = 2^{n+1} - 1$ and $s_2(b) = 1$. So by (31) and (45), Lemma 4.1 (iii) follows immediately.

This completes the proof of Lemma 4.1. \square

We are now in the position to show Theorem 1.4.

Proof of Theorem 1.4. By Lemma 2.6, we get that

$$S(c2^m + b2^{n+1} + 2^n, b2^{n+2} + a) = \sum_{j=1}^{b2^{n+1}+2^n} \sum_{i=0}^j h(i, j) = \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 + \Delta, \quad (46)$$

where $h(i, j)$ and Δ_l ($l = 1, 2, 3, 4$) are defined as in (30) and (31), respectively, and

$$\Delta := \sum_{j=2^{n+1}}^{b2^{n+1}} \sum_{i=0}^j h(i, j). \quad (47)$$

First we deal with the 2-adic valuation of $h(i, j)$ with $2^{n+1} \leq j \leq b2^{n+1}$ and $0 \leq i \leq j$. Let $j = j_1 2^{n+1} + j_2$ for some integers $1 \leq j_1 \leq b$ and $0 \leq j_2 < 2^{n+1}$.

If $j_2 = 0$, then $j = j_1 2^{n+1}$. So by Lemmas 2.2-2.4 and (30), we have

$$\begin{aligned} v_2(h(i, j)) &\geq v_2\left(\frac{(b2^{n+2} + a - i)!}{(b2^{n+2} + a - j)!}\right) + v_2(S(c2^m, b2^{n+2} + a - i)) \\ &\geq j - i + s_2(b2^{n+2} + a - j) - s_2(b2^{n+2} + a - i) + s_2(b2^{n+2} + a - i) - 1 \\ &\geq s_2(b2^{n+2} + a - j) - 1 \\ &= s_2((2b - j_1)2^{n+1} + a) - 1 \\ &\geq s_2(a) \end{aligned} \quad (48)$$

since $s_2(2b - j_1) \geq 1$ and $a < 2^{n+1}$.

If $0 < j_2 < 2^{n+1}$, then $1 \leq j_1 < b$. By Theorem 1.1 we have

$$v_2(S(b2^{n+1} + 2^n, j)) = v_2(S(b2^{n+1} + 2^n, j_1 2^{n+1} + j_2)) \geq s_2(j_2) - 1. \quad (49)$$

Thus by Lemmas 2.2-2.3, (30), (32) and (49) we deduce

$$\begin{aligned} v_2(h(i, j)) &\geq j - i + s_2(b2^{n+2} + a - j) + s_2(j_2) - 2 \\ &\geq s_2(j_2) + s_2(b2^{n+2} - j_1 2^{n+1} + a - j_2) - 2 \\ &\geq s_2((2b - j_1)2^{n+1} + a) - 2 \\ &= s_2(2b - j_1) + s_2(a) - 2. \end{aligned} \quad (50)$$

Let A_t be defined as in (17). Then $\mathbb{Z}^+ = \bigcup_{t=1}^{\infty} A_t$. We prove Theorem 1.4 by induction on t . First we consider that the case $t = 1$. Let $b \in A_1$. Then $s_2(b) = 1$. So $s_2(2b - j_1) \geq 2$ if $j_1 < b$. Thus by (50) we have that $v_2(h(i, j)) \geq s_2(a)$ if $0 < j_2 < 2^{n+1}$. Furthermore, by (47) and (48) we get

$$v_2(\Delta) \geq s_2(a). \quad (51)$$

By Lemma 4.1 (iv), (46) and (51), Theorem 1.4 for the case $s_2(b) = 1$ follows immediately. That is, Theorem 1.4 holds if $t = 1$.

Now let $t \geq 2$. Assume that Theorem 1.4 is true for any integers $b \in A_1 \cup \dots \cup A_{t-1}$. In what follows we prove Theorem 1.4 is true for the case t , namely, for the case that $b \in A_t$.

For $b \in A_t$, let $b = 2^{r_1} + 2^{r_2} + \dots + 2^{r_t}$ be the 2-adic expansion of b , where $r_1 > r_2 > \dots > r_t$. We claim that if $1 \leq j_1 < b$, then $s_2(2b - j_1) = 1$ if and only if $b = 2^{r_1} + \frac{j_1}{2}$. We first notice that if $1 \leq j_1 < b$, then

$$2^{r_1+2} > 2b > 2b - j_1 > 2^{r_1}.$$

So $s_2(2b - j_1) = 1$ if and only if $2b - j_1 = 2^{r_1+1}$, i.e., $b = 2^{r_1} + \frac{j_1}{2}$. The claim is proved. In the following we handle Δ . If $s_2(2b - j_1) \geq 2$, then by (50) we derive that

$$v_2(h(i, j)) \geq s_2(a). \quad (52)$$

Now let $s_2(2b - j_1) = 1$. Then by the claim we have $b = 2^{r_1} + \frac{j_1}{2}$. It follows that

$$S(b2^{n+1} + 2^n, j_1 2^{n+1} + j_2) = S\left(2^{r_1+n+1} + \frac{j_1}{2} 2^{n+1} + 2^n, \frac{j_1}{2} 2^{n+2} + j_2\right). \quad (53)$$

Since $2b - j_1 = 2^{r_1+1}$, we get $j_1 = 2^{r_2+1} + \dots + 2^{r_t+1}$, which implies that $s_2(\frac{j_1}{2}) = t - 1$ and so $\frac{j_1}{2} \in A_{t-1}$. Hence the inductive hypothesis applied to (53) we know that

$$v_2(S(b2^{n+1} + 2^n, j_1 2^{n+1} + j_2)) \begin{cases} = s_2(j_2) - 1 = n, & \text{if } j_2 = 2^{n+1} - 1, \\ \geq s_2(j_2), & \text{if } j_2 < 2^{n+1} - 1. \end{cases} \quad (54)$$

If $j_2 < 2^{n+1} - 1$, then by Lemmas 2.2-2.4, (30), (32) and (54) we have

$$\begin{aligned} v_2(h(i, j)) &\geq j - i + s_2(b2^{n+2} + a - j) + s_2(j_2) - 1 \\ &\geq s_2(b2^{n+2} - j_1 2^{n+1} + a - j_2) + s_2(j_2) - 1 \\ &\geq s_2((2b - j_1)2^{n+1} + a) - 1 \\ &= s_2(a). \end{aligned} \quad (55)$$

If $j_2 = 2^{n+1} - 1$, then we have

$$b2^{n+2} + a - j = (2b - j_1)2^{n+1} + a - j_2 = 2^{n+2+r_1} + a - j_2 \leq 2^{n+r_1+2} \leq 2^m \quad (56)$$

since $m \geq n + 2 + \lfloor \log_2 b \rfloor = n + 2 + r_1$. Then by Lemma 2.3 and (56) we have

$$S(c2^m, b2^{n+2} + a - j) = s_2(b2^{n+2} + a - j) - 1. \quad (57)$$

It then follows from Lemmas 2.2-2.4, (30), (32), (54) and (57) that

$$\begin{aligned} v_2(h(i, j)) &\geq s_2(b2^{n+2} + a - j) - 1 + s_2(j_2) - 1 + j - i \\ &\geq s_2(b2^{n+2} - j_1 2^{n+1} + a - j_2) + s_2(j_2) - 2 \\ &= s_2((2b - j_1)2^{n+1} + a - 2^{n+1} + 1) + n - 1 \\ &\geq n \end{aligned} \quad (58)$$

with equality holding if and only if $j = i$ and $a = 2^{n+1} - 1$.

Finally, by (48), (55), (58) and (52) we obtain that if $b \in A_t$, then

$$v_2(\Delta) \begin{cases} = n & \text{if } a = 2^{n+1} - 1, \\ \geq s_2(a) & \text{if } a < 2^{n+1} - 1. \end{cases} \quad (59)$$

Hence by Lemma 4.1 (iv), (46) and (59) we conclude that Theorem 1.4 is true if $b \in A_t$.

The proof of Theorem 1.4 is complete. \square

5 Proof of Theorem 1.3

For any positive integer k , we define $\theta(k)$ to be the largest integer l with $1 \leq l \leq s_2(k)$ such that $\{m_l, m_{l-1}, \dots, m_1\}$ is a set of consecutive integers, where $k = 2^{m_1} + 2^{m_2} + \dots + 2^{m_{s_2(k)}}$ is the 2-adic expansion of k and $m_1 > m_2 > \dots > m_{s_2(k)}$. Then $\lceil \log_2 k \rceil = m_1 + 1$. We have the following result.

Lemma 5.1 *Let $n, k, a, c \in \mathbb{Z}^+$ be such that $3 \leq k \leq 2^n$ and $1 \leq a \leq \lceil \frac{k}{2} \rceil - 1$. Suppose that k is neither a power of 2 nor a power 2 minus 1. Then we have*

$$v_2(S(c2^n - a, k - 2a)) = s_2(k) - \lceil \log_2 k \rceil + v_2(a)$$

if either $a = \sum_{i=m_{\theta(k)}}^{m_1} 2^{i-1}$ with $\theta(k) < s_2(k)$ or $a = \sum_{i=m_{\theta(k)+1}}^{m_1} 2^{i-1}$ with $\theta(k) = s_2(k)$, and

$$v_2(S(c2^n - a, k - 2a)) > s_2(k) - \lceil \log_2 k \rceil + v_2(a)$$

otherwise.

Proof. First, we can write

$$k = \sum_{i=m_{\theta(k)}}^{m_1} 2^i + \sum_{j=\theta(k)+1}^{s_2(k)} 2^{m_j}. \quad (60)$$

Note that the second sum in (60) vanishes if $\theta(k) = s_2(k)$. Obviously, $m_1 = m_l + l - 1$ if $1 \leq l \leq \theta(k)$ and $m_{\theta(k)} \geq m_{\theta(k)+1} + 2$ if $\theta(k) < s_2(k)$.

If $a = \sum_{i=m_{\theta(k)}}^{m_1} 2^{i-1}$ with $\theta(k) < s_2(k)$, then by (60) we infer that $k - 2a = \sum_{j=\theta(k)+1}^{s_2(k)} 2^{m_j}$ and $v_2(c2^n - a) = v_2(a) = m_{\theta(k)} - 1 = m_1 - \theta(k)$. It then follows from $\lceil \log_2 k \rceil = m_1 + 1$ that

$$s_2(k - 2a) = s_2(k) - \theta(k) \quad (61)$$

and

$$\theta(k) = m_1 - v_2(a) = \lceil \log_2 k \rceil - 1 - v_2(a). \quad (62)$$

Since $m_{\theta(k)} \geq m_{\theta(k)+1} + 2$, we have $k - 2a < 2^{m_{\theta(k)}-1} = 2^{v_2(c2^n - a)}$. It then follows from Lemma 2.3, (61) and (62) that

$$v_2(S(c2^n - a, k - 2a)) = s_2(k - 2a) - 1 = s_2(k) - \lceil \log_2 k \rceil + v_2(a)$$

as required. Hence Lemma 5.1 is proved if $a = \sum_{i=m_{\theta(k)}}^{m_1} 2^{i-1}$ with $\theta(k) < s_2(k)$.

If $a = \sum_{i=m_{\theta(k)+1}}^{m_1} 2^{i-1}$ with $\theta(k) = s_2(k)$, then by (60) we deduce that $k - 2a = 2^{m_{\theta(k)}}$ and $v_2(c2^n - a) = v_2(a) = m_{\theta(k)} = m_1 + 1 - \theta(k) = \lceil \log_2 k \rceil - s_2(k)$ since $\lceil \log_2 k \rceil = m_1 + 1$. So $s_2(k) - \lceil \log_2 k \rceil + v_2(a) = 0$. It then follows from Lemma 2.3 that

$$v_2(S(c2^n - a, k - 2a)) = s_2(2^{m_{\theta(k)}}) - 1 = 0 = s_2(k) - \lceil \log_2 k \rceil + v_2(a).$$

Thus Lemma 5.1 is proved if $a = \sum_{i=m_{\theta(k)+1}}^{m_1} 2^{i-1}$ with $\theta(k) = s_2(k)$.

Now we treat the remaining case that neither $a = \sum_{i=m_{\theta(k)}}^{m_1} 2^{i-1}$ with $\theta(k) < s_2(k)$ nor $a = \sum_{i=m_{\theta(k)+1}}^{m_1} 2^{i-1}$ with $\theta(k) = s_2(k)$. In this remaining case, we claim that

$$v_2(S(c2^n - a, k - 2a)) \geq s_2(k) - m_1 + v_2(a). \quad (63)$$

From the claim (63) and noting that $\lceil \log_2 k \rceil = m_1 + 1$, we derive that

$$v_2(S(c2^n - a, k - 2a)) > s_2(k) - \lceil \log_2 k \rceil + v_2(a).$$

So Lemma 5.1 holds for the remaining case that neither $a = \sum_{i=m_{\theta(k)}}^{m_1} 2^{i-1}$ with $\theta(k) < s_2(k)$ nor $a = \sum_{i=m_{\theta(k)+1}}^{m_1} 2^{i-1}$ with $\theta(k) = s_2(k)$. Thus we need only to prove that the claim (63) is true, which will be done in what follows.

If $v_2(a) < m_{s_2(k)}$, then $s_2(k) - (m_1 - v_2(a)) \leq s_2(k) - (m_1 - m_{s_2(k)} + 1) \leq 0$ since $s_2(k) \leq m_1 - m_{s_2(k)} + 1$. This concludes that the claim (63) is true if $v_2(a) < m_{s_2(k)}$.

If $m_{s_2(k)} \leq v_2(a) < m_{\theta(k)} - 1$, then $\theta(k) < s_2(k)$ and there is exactly one integer t with $\theta(k) < t \leq s_2(k)$ such that $m_t \leq v_2(a) < m_{t-1}$. Then $\{v_2(a), m_{t-1}, \dots, m_{\theta(k)}, \dots, m_1\}$ is not consisting of consecutive integers. This implies that $s_2(2^{m_1} + \dots + 2^{m_{t-1}} + 2^{m_t}) = s_2(2^{m_1} + \dots + 2^{m_{t-1}} + 2^{v_2(a)}) \leq m_1 - v_2(a)$. Therefore

$$\begin{aligned} s_2(2^{m_t} + \dots + 2^{m_{s_2(k)}}) &= s_2(k) - s_2(2^{m_1} + \dots + 2^{m_{t-1}} + 2^{m_t}) + 1 \\ &\geq s_2(k) - (m_1 - v_2(a)) + 1. \end{aligned} \quad (64)$$

Since $v_2(c2^n - a) = v_2(a)$ and $m_t \leq v_2(a) < m_{t-1}$, one may write $c2^n - a = c_1 2^{v_2(a)} + c_2 2^{v_2(a)+1} + 2^{m_t} + \dots + 2^{m_{s_2(k)}}$ with c_1 and c_2 being integers. Then by Theorem 1.1 and (64) we deduce that

$$\begin{aligned} v_2(S(c2^n - a, k - 2a)) &= v_2(S(c_1 2^{v_2(a)}, c_2 2^{v_2(a)+1} + 2^{m_t} + \dots + 2^{m_{s_2(k)}})) \\ &\geq s_2(2^{m_t} + \dots + 2^{m_{s_2(k)}}) - 1 \\ &\geq s_2(k) - m_1 + v_2(a) \end{aligned}$$

as desired. Hence the claim (63) holds if $m_{s_2(k)} \leq v_2(a) < m_{\theta(k)} - 1$.

If $m_{\theta(k)} - 1 \leq v_2(a) \leq m_1 - 1$, then by (60) we can write

$$k - 2a = \sum_{i=v_2(a)+1}^{m_1} 2^i - 2a + u = b2^{v_2(a)+2} + u \quad (65)$$

and

$$c2^n - a = c_3 2^{m_1} + \sum_{i=v_2(a)}^{m_1-1} 2^i + 2^{v_2(a)} - a = c_3 2^{m_1} + b2^{v_2(a)+1} + 2^{v_2(a)}, \quad (66)$$

where $c_3 \in \mathbb{Z}^+$ and u and b are defined as follows:

$$u =: \sum_{i=m_{\theta(k)}}^{v_2(a)} 2^i + \sum_{j=\theta(k)+1}^{s_2(k)} 2^{m_j}, \quad b =: \left(\sum_{i=v_2(a)}^{m_1-1} 2^i - a \right) / 2^{v_2(a)+1}.$$

Since k is not a power of 2 minus 1, one has $m_{\theta(k)} \geq 1$ if $\theta(k) = s_2(k)$. It follows that $\sum_{i=m_{\theta(k)}}^{m_1} 2^{i-1} = \lceil \frac{k}{2} \rceil$ if $\theta(k) = s_2(k)$. Since $1 \leq a < \lceil \frac{k}{2} \rceil$, we have $a \neq \sum_{i=m_{\theta(k)}}^{m_1} 2^{i-1}$ if $\theta(k) = s_2(k)$. Note that the assumption tells us that $a \neq \sum_{i=m_{\theta(k)}}^{m_1} 2^{i-1}$ if $\theta(k) < s_2(k)$. Then from $m_{\theta(k)} - 1 \leq v_2(a)$ we derive that b is a positive integer. Since k is not a power of 2 minus 1, we have $\sum_{i=m_{\theta(k)}}^{v_2(a)} 2^i < 2^{v_2(a)+1} - 1$. It then follows that if $a \neq \sum_{i=m_{\theta(k)+1}}^{m_1} 2^{i-1}$ with $\theta(k) = s_2(k)$, then $0 < u = \sum_{i=m_{\theta(k)}}^{v_2(a)} 2^i < 2^{v_2(a)+1} - 1$. Notice that $m_{\theta(k)} \geq m_{\theta(k)+1} + 2$ if $\theta(k) < s_2(k)$. Thus $0 < u < 2^{v_2(a)+1} - 1$ if $\theta(k) < s_2(k)$. Meanwhile, by (60) we get that

$$s_2(u) = s_2(k) - s_2\left(\sum_{i=v_2(a)+1}^{m_1} 2^i\right) = s_2(k) - m_1 + v_2(a). \quad (67)$$

Therefore by (65)-(67) and Theorem 1.4, we derive that

$$\begin{aligned} v_2(S(c2^n - a, k - 2a)) &= S(c_3 2^{m_1} + b2^{v_2(a)+1} + 2^{v_2(a)}, b2^{2^{v_2(a)+2}} + u) \\ &\geq s_2(u) = s_2(k) - m_1 + v_2(a) \end{aligned}$$

as required. The claim (63) is true if $m_{\theta(k)} - 1 \leq v_2(a) \leq m_1 - 1$. Thus the claim (63) is proved.

This completes the proof of Lemma 5.1. \square

We are now in the position to show Theorem 1.3.

Proof of Theorem 1.3. Suppose that (2) is true. Then using (2) with $a = 2c$ and $b = c$, we can easily derive that (3) holds. So we only need to show that (2) is true, which will be done in the following.

To prove (2), use (4) and (5) with $p = 2$, $m = (2b - a)2^n$, $v = 1$ and n replaced by $(a - b)2^n$, and consider the coefficients of x^k :

$$\begin{aligned}
S(a2^n, k) &\equiv \sum_{j=0}^{(a-b)2^n} \binom{(a-b)2^n}{j} S(j + (2b - a)2^n, k - 2((a - b)2^n - j)) \\
&= S(b2^n, k) + \sum_{j=(a-b)^n - \lceil \frac{k}{2} \rceil + 1}^{(a-b)2^n - 1} \binom{(a-b)2^n}{j} S(j + (2b - a)2^n, k - 2((a - b)2^n - j)) \\
&= S(b2^n, k) + \sum_{i=1}^{\lceil \frac{k}{2} \rceil - 1} \binom{(a-b)2^n}{i} S(b2^n - i, k - 2i) \pmod{2^{n+v_2(a-b)}}. \tag{68}
\end{aligned}$$

It then follows from (68) that

$$S(a2^n, k) - S(b2^n, k) \equiv \sum_{i=1}^{\lceil \frac{k}{2} \rceil - 1} \binom{(a-b)2^n}{i} S(b2^n - i, k - 2i) \pmod{2^{n+v_2(a-b)}}. \tag{69}$$

In what follows we discuss the 2-adic valuation of a general term of (69) with $1 \leq i \leq \lceil \frac{k}{2} \rceil - 1$. Let $a - b = c_0 2^{v_2(a-b)}$ with $c_0 \geq 1$ being odd. We first notice that $i \leq \lceil \frac{k}{2} \rceil - 1 < 2^n$. So by Lemma 2.2 and 2.11 we get that

$$\begin{aligned}
&v_2\left(\binom{(a-b)2^n}{i} S(b2^n - i, k - 2i)\right) \\
&= s_2(i) + s_2(c_0 2^{n+v_2(a-b)} - i) - s_2(c_0 2^{n+v_2(a-b)}) + v_2(S(b2^n - i, k - 2i)) \\
&= n + v_2(a - b) - v_2(i) + v_2(S(b2^n - i, k - 2i)). \tag{70}
\end{aligned}$$

We consider the following two case.

Case 1. $s_2(k) = 1$. Then one may write $k = 2^m$. If $m = 2$, then by (1) we have

$$\begin{aligned}
&v_2(S(a2^n, 4) - S(b2^n, 4)) \\
&= v_2\left(\frac{1}{6}(4^{a2^n-1} - 3^{a2^n} + 3 \cdot 2^{a2^n-1} - 1) - \frac{1}{6}(4^{b2^n-1} - 3^{b2^n} + 3 \cdot 2^{b2^n-1} - 1)\right) \\
&= v_2(3^{b2^n}(3^{(a-b)2^n} - 1)) - 1
\end{aligned}$$

By Lemma 2.12 we have $v_2(3^{(a-b)2^n} - 1) = n + v_2(a - b) + 2$. Thus we get that

$$v_2(S(a2^n, 4) - S(b2^n, 4)) = n + v_2(a - b) - \lceil \log_2 4 \rceil + s_2(4) + \delta(4)$$

since $\delta(4) = 2$. Namely, Theorem 1.3 holds if $m = 2$.

Now let $m \geq 3$. So $1 \leq i \leq 2^{m-1} - 1$. If $i = 2^{m-2}$, then by Lemma 2.5

$$v_2(S(b2^n - i, 2^m - 2i)) = v_2(S(b2^n - 2^{m-2}, 2^{m-1})) = 0. \tag{71}$$

Thus by (70) and (71) we obtain that

$$v_2\left(\binom{(a-b)2^n}{i}S(b2^n-i, 2^m-2i)\right) = n + v_2(a-b) - (m-2). \quad (72)$$

If $i \neq 2^{m-2}$, then $v_2(i) < 2^{m-2}$ since $i \leq 2^{m-1} - 1$. It then follows from (71) that

$$\begin{aligned} v_2\left(\binom{(a-b)2^n}{i}S(b2^n-i, 2^m-2i)\right) &> n + v_2(a-b) - (m-2) + v_2(S(b2^n-i, 2^m-2i)) \\ &\geq n + v_2(a-b) - (m-2). \end{aligned} \quad (73)$$

Hence by (69), (72) and (73) we derive that

$$\begin{aligned} S(a2^n, 2^m) - S(b2^n, 2^m) &= \min_{1 \leq i \leq 2^{m-1}-1} \left\{ v_2\left(\binom{(a-b)2^n}{i}S(b2^n-i, 2^m-2i)\right) \right\} \\ &= n + v_2(a-b) - m + 2 \\ &= n + v_2(a-b) - \lceil \log_2 2^m \rceil + s_2(2^m) + \delta(2^m) \end{aligned}$$

since $\delta(2^m) = 1$. So (2) is true if $s_2(k) = 1$.

Case 2. $s_2(k) \geq 2$. Since k is neither a power of 2 nor a power 2 minus 1 and $1 \leq i \leq \lceil \frac{k}{2} \rceil - 1$, by Lemma 5.1, (69) and (70) we obtain that

$$\begin{aligned} S(a2^n, k) - S(b2^n, k) &= \min_{1 \leq i \leq \lceil \frac{k}{2} \rceil - 1} \left\{ v_2\left(\binom{(a-b)2^n}{i}S(b2^n-i, k-2i)\right) \right\} \\ &= n + v_2(a-b) - \lceil \log_2 k \rceil + s_2(k) \\ &= n + v_2(a-b) - \lceil \log_2 k \rceil + s_2(k) + \delta(k). \end{aligned}$$

since $\delta(k) = 0$. Hence (2) holds in this case.

The proof of Theorem 1.3 is complete. □

References

- [1] T. Agoh and K. Dicher, Generalized convolution identities for Stirling numbers of the second kind, *Integers* **8** (2008) #A25, 7pp.
- [2] T. Amdeberhan, D. Manna and V. Moll, The 2-adic valuation of Stirling numbers, *Experimental Math.* **17** (2008), 69-82.
- [3] M. Bendersky and D.M. Davis, 2-primary v_1 -periodic homotopy groups of $SU(n)$, *Amer. J. Math.* **114**(1991) 529-544.
- [4] L. Carlitz, Congruences for generalized Bell and Stirling numbers, *Duke Math. J.* **22**(1955), 193-205.
- [5] O-Y. Chan and D.V. Manna, Congruences for Stirling numbers of the Second kind, *Contemporary Math.* **517**(2010), 97-111.
- [6] F. Clarke, Hensel's lemma and the divisibility by primes of Stirling-like numbers, *J. Number Theory* **52** (1995), 69-84.
- [7] M.C. Crabb and K. Knapp, The Hurewicz map on stunted complex projective spaces, *Amer. J. Math.* **110** (1988), 783-809.

- [8] D.M. Davis, Divisibility by 2 of Stirling-like numbers, *Proc. Amer. Math. Soc.* **110** (1990), 597-600.
- [9] D.M. Davis, v_1 -periodic homotopy groups of $SU(n)$ at odd primes, *Proc. London Math. Soc.* **43** (1991), 529-541.
- [10] D.M. Davis, Divisibility by 2 and 3 of certain Stirling numbers, *Integers* **8** (2008) A56, 25pp.
- [11] D.M. Davis and K. Potocka, 2-primary v_1 -periodic homotopy groups of $SU(n)$ revisited, *Forum Math.* **19** (2007), 783-822.
- [12] S. Hong, J. Zhao and W. Zhao, The 2-adic valuations of Stirling numbers of the second kind, *International J. Number Theory* **8** (2012), 1057-1066.
- [13] A. Junod, Congruences pour les polynômes et nombres de Bell, *Bull. Belg. Math. Soc.* **9** (2002), 503-509.
- [14] E.E. Kummer, Über die Ergänzungssätze zu den allgemeinen Reciprocitätsgesetzen, *J. Reine Angew. Math. Theory* **44** (1852), 93-146.
- [15] Y.H. Kwong, Minimum periods of $S(n, k)$ modulo M , *Fibonacci Quart.* **27** (1989), 217-221.
- [16] T. Lengyel, On the divisibility by 2 of Stirling numbers of the second kind, *Fibonacci Quart.* **32** (1994), 194-201.
- [17] T. Lengyel, On the 2-adic order of Stirling numbers of the second kind and their differences, *DMTCS Proc. AK*, 2009: 561-572.
- [18] T. Lengyel, Alternative proofs on the 2-adic order of Stirling numbers of the Second kind, *Integers* **10** (2010) #A38, 453-463.
- [19] A.T. Lundell, Generalized e -invariants and the numbers of James, *Quart. J. Math. Oxford* **25** (1974), 427-440.
- [20] A.T. Lundell, A divisibility property for Stirling numbers, *J. Number Theory* **10** (1978), 35-54.
- [21] S. De Wannemacker, On the 2-adic orders of Stirling numbers of the second kind, *Integers* **5** (2005) #A21, 7pp.
- [22] P.T. Young, Congruences for degenerate number sequence, *Discrete Math.* **270** (2003), 279-289.